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THE USE OF THE TETRACHORIC SERIES FOR EVALUATING MULTIVARIATE N--ETC(U)

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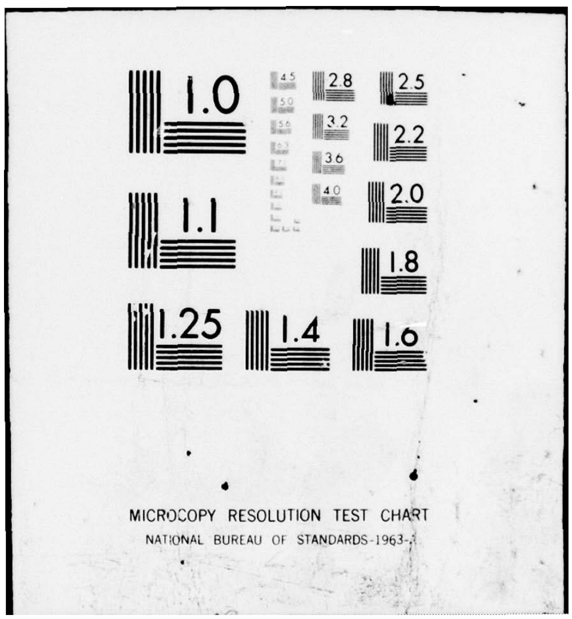
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(6) # THE USE OF THE TETRACHORIC SERIES FOR EVALUATING MULTIVARIATE NORMAL PROBABILITIES.

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THE USE OF THE TETRACHORIC SERIES FOR EVALUATING MULTIVARIATE NORMAL PROBABILITIES

Bernard Harris and Andrew P. Soms

Technical Summary Report #2015
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ABSTRACT

The tetrachoric series is a technique for evaluating multivariate normal probabilities frequently cited in the statistical literature. In this paper we have examined the convergence properties of the tetrachoric series and have established the following.

abs. val. (rho sub ij) For orthant probabilities, the tetrachoric series converges if $|\rho_{ij}| < 1/(k-1)$, $1 \leq i < j \leq k$, where ρ_{ij} (rho sub ij) are the correlation coefficients of a k-variate normal distribution. The tetrachoric series for orthant probabilities diverges whenever k is even and $\rho_{ij} > 1/(k-1)$ or k is odd and $\rho_{ij} > 1/(k-2)$, $1 \leq i < j \leq k$.

Other specific results concerning the convergence or divergence of this series are also given.

The principal point is that the assertion that the tetrachoric series converges for all $k \geq 2$ and all ρ_{ij} (rho sub ij) such that the correlation matrix is positive definite is false.

AMS(MOS) Subject Classification: Primary 62H05

Key Words: Tetrachoric series, Multivariate normal distribution

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SIGNIFICANCE AND EXPLANATION

A standard technique for evaluating multivariate normal probabilities, widely cited in the literature, is known as the tetrachoric series.

In this report, it is shown that this technique is defective in that the series will sometimes diverge.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE USE OF THE TETRACHORIC SERIES
FOR EVALUATING MULTIVARIATE NORMAL PROBABILITIES

Bernard Harris and Andrew P. Soms

1. INTRODUCTION AND SUMMARY

Let $\tilde{X} = (X_1, X_2, \ldots, X_k)$, $k \geq 2$, be a normally distributed random vector with zero means and unit variances, that is, $\tilde{X}$ has the probability density function

$$f_{\tilde{X}}(x_1, x_2, \ldots, x_k) = (2\pi)^{-k/2} |R|^{-\frac{1}{2}} e^{-\frac{1}{2}\tilde{x}^T R^{-1} \tilde{x}} , \qquad (1.1)$$

where $\tilde{x} = (x_1, x_2, \ldots, x_k)^T$, A^T denotes the transpose of the matrix A, and R, the correlation, matrix is a $k \times k$ positive definite symmetric matrix with elements ρ_{ij} and $\rho_{ii} = 1$, $i = 1, 2, \ldots, k$. Further, let

$$\phi(x) = (2\pi)^{-\frac{1}{2}} e^{-x^2/2} ,$$

the standard normal probability density function, and let

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt.$$

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In addition, let

$$H_{-1}(x) = (1-\Phi(x))/\phi(x) \tag{1.2}$$

and for $i \geq 0$, let

$$H_i(x) = e^{x^2/2}(-1)^i \frac{d^i}{dx^i}(e^{-x^2/2}). \tag{1.3}$$

For $i \geq 0$, $H_i(x)$ are the Hermite polynomials.

In order to evaluate

$$P\{X_1>h_1, X_2>h_2, \ldots, X_k>h_k\} = \int_{h_k}^{\infty} \cdots \int_{h_2}^{\infty} \int_{h_1}^{\infty} f_{\underset{\sim}{X}}(x_1, x_2, \ldots, x_k)\,dx_1 dx_2 \ldots dx_k, \tag{1.4}$$

M.G. Kendall [6] proposed writing

$$P\{X_1>h_1, X_2>h_2, \ldots, X_k>h_k\} =$$

$$\sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} \prod_{1\le i<j\le k} \frac{\rho_{ij}^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k} H_{n_i-1}(h_i)\phi(h_i), \tag{1.5}$$

where $n_i = \sum_{j:i<j} n_{ij} + \sum_{j:j<i} n_{ji}$.

The right hand side of (1.5) is called the tetrachoric series. The particular case obtained by ordering the terms of (1.5) with respect to increasing values of Σn_{ij}, namely

$$\sum_{N=0}^{\infty} \sum_{n_{12}} \cdots \sum_{n_{k-1,k}} \prod_{1\le i<j\le k} \frac{\rho_{ij}^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k} H_{n_i-1}(h_i)\phi(h_i), \tag{1.6}$$

where $n_{ij} \geq 0$ and $\Sigma n_{ij} = N$, is known as the Kibble series or the tetrachoric series with Kibble ordering, since W.F. Kibble [9] obtained the corresponding series for the multivariate normal density function. Note that (1.6) is the result that would be obtained upon integrating the series given in [9] term-by-term. Consequently, to avoid ambiguity, we will refer to the series obtained by Kibble in [9] as the Kibble series and we will call the corresponding series for the multivariate normal distribution the tetrachoric series with Kibble ordering.

In the sequel, to simplify notation, the right hand side of (1.5) will be denoted by $T_k(h_1,h_2,\ldots h_k)$ and $T_k(0,0,\ldots,0)$ will be denoted by T_k. Similarly, the right hand side of (1.6) will be denoted by $T_k^*(h_1,h_2,\ldots,h_k)$ and T_k^*. Also, sums or products on (i,j) with $1 \leq i < j \leq k$ will be indexed by $i < j$.

The tetrachoric series is widely employed in applications of multivariate analysis and is quoted in many standard books on statistics and in many papers as a suggested method for evaluating (1.4), often with the comment that convergence may be slow unless $|\rho_{ij}|$ is small for all i,j, $i \neq j$. Specifically, the reader is referred to T.W. Anderson [1], page 19, where the tetrachoric series is suggested as a way of evaluating multivariate normal probabilities. In R.E. Barlow, D.J. Bartholomew, J.M. Bremner, and H.D. Brunk [2], page 137, the tetrachoric series is mentioned as one of several possible ways to evaluate orthant probabilities. They comment that the convergence is not "fast enough for practical use unless the ρ_{ij}'s are small". R.L. Plackett [12] comments that "although the tetrachoric series will always converge, it does so very slowly when the absolute values of the correlation coefficients are near unity". The Kibble and tetrachoric series are described in connection with orthant probabilities in N.L. Johnson and S. Kotz [5], pages 44-46, where they note that "these series converge very slowly unless all the ρ_{ij}'s are small". M.G. Kendall and A. Stuart [8] describe the tetrachoric series on pages 352-353 and assert that the series for the orthant probabilities always converges, but only slowly if the ρ_{ij} are not small. G.P. Steck [13] described the Kendall technique noting that the resulting series converges slowly when the ρ_{ij} are large. S.S. Gupta [4] discussed both the results of Kendall and Kibble for the trivariate normal distribution, remarking that the series converge only very slowly for high values of $|\rho_{ij}|$. In [7], M.G. Kendall employed the tetrachoric series for orthant probabilities to study the distribution of upruns (sequences of increasing observations) in a time series. For this purpose, he specifically gives the terms of the tetrachoric series for the orthant probabilities for $k = 2,3,4$. He says that "the expressions are not as difficult as they look" and that "they converge fairly quickly for damped autoregressive series". Kendall says further that "they are, he thinks, amenable to calculation".

In this paper, we have reexamined the convergence of the tetrachoric series and have ascertained that under specified conditions on the correlation matrix R, the tetrachoric series will in fact diverge.

The development in this paper is as follows. In section two, we discuss the Kibble series for the multivariate normal density function, which is an intermediate step in arriving at the tetrachoric series. This will permit us to subsequently indicate the difficulty with the tetrachoric series. A sufficient condition for the convergence of the tetrachoric series is given in section three. The third section is also devoted to illustrations of the non-convergence of the tetrachoric series.

2. The Kibble Series for the Multivariate Normal Density Function

F.G. Mehler [10] derived a series expansion for the density function of the bivariate normal distribution. This was subsequently extended to the general k-variate density (1.1) by W.F. Kibble [9]. Unfortunately, Kibble's paper contains some defects and as we will show, the Kibble series does not necessarily converge. Since this is an intermediate step in arriving at the tetrachoric series, it is desirable to rederive the Kibble series, which will permit us to obtain both a correct result as well as to indicate the error in Kibble's paper. Accordingly, we first establish the following theorem.

THEOREM 1. For $-1/(k-1) < \rho_{ij} < 1/(k-1)$, $1 \leq i < j \leq k$,

$$f_{\tilde{X}}(x_1,x_2,\ldots,x_k) = \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} \prod_{i<j} \frac{\rho_{ij}^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k} H_{n_i}(x_i)\phi(x_i). \quad (2.1)$$

This series is known as the Kibble series and converges absolutely if the ρ_{ij} satisfy the above restrictions.

Proof. Let $\psi(t_1,t_2,\ldots,t_k) = e^{-\frac{1}{2}\tilde{t}^T R\tilde{t}}$, $\tilde{t} = (t_1,t_2,\ldots,t_k)^T$.

$\psi(t_1,t_2,\ldots,t_k)$ is the characteristic function of the k-variate normal distribution. Then, from the inversion theorem,

$$f_{\tilde{X}}(x_1,x_2,\ldots,x_k) = (2\pi)^{-k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\tilde{t}^T\tilde{x}}\, \psi(t_1,t_2,\ldots,t_k)dt_1dt_2\ldots dt_k. \quad (2.2)$$

Writing

$$\psi(\tilde{t}) = e^{-\frac{1}{2}\tilde{t}^T\tilde{t} - \sum_{i<j}\rho_{ij}t_it_j} = e^{-\frac{1}{2}\tilde{t}^T\tilde{t}} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \left(\sum_{i<j}\rho_{ij}t_it_j\right)^N \quad (2.3)$$

and expanding the last factor in a multinomial series, we obtain

$$\psi(\tilde{t}) = e^{-\frac{1}{2}\tilde{t}^T\tilde{t}} \sum_{N=0}^{\infty} (-1)^N \sum \frac{\rho_{12}^{n_{12}} \cdots \rho_{k-1,k}^{n_{k-1,k}} t_1^{n_1} \cdots t_k^{n_k}}{n_{12}! \cdots n_{k-1,k}!} , \tag{2.4}$$

where the inner sum is over $n_{ij} \geq 0$ with $\Sigma n_{ij} = N$. A rearrangement of terms yields

$$\psi(\tilde{t}) = e^{-\frac{1}{2}\tilde{t}^T\tilde{t}} \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} \prod_{i<j} \frac{(-\rho_{ij})^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k} t_i^{n_i} . \tag{2.5}$$

For $|\rho_{ij}| < \rho < 1/(k-1)$, $1 \leq i < j \leq k$, let

$$g(\tilde{t}) = e^{\rho \sum_{i<j} |t_i t_j|} . \tag{2.6}$$

Then

$$|e^{-i\tilde{t}^T\tilde{x} - \frac{1}{2}\tilde{t}^T t} g(\tilde{t})| = e^{-\frac{1}{2}\tilde{t}^T\tilde{t}} g(\tilde{t}) \tag{2.7}$$

obviously dominates the partial sums in (2.4) and (2.5) and as the following argument shows, (2.7) is integrable.

Note that

$$(2\pi)^{-k/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\tilde{t}^T\tilde{t}} g(\tilde{t})\, dt_1 dt_2 \ldots dt_k = 2^k |\Sigma|^{\frac{1}{2}} P\{X_1>0,\ldots,X_k>0\}, \tag{2.8}$$

where $(X_1, X_2, \ldots, X_k)$ has the multivariate normal distribution with $\Sigma^{-1} = (\sigma^{ij})$, $\sigma^{ii} = 1$, $\sigma^{ij} = -\rho$, $i \neq j$ and the hypothesis $|\rho| < 1/(k-1)$ insures that Σ is positive definite. Thus, we can substitute (2.4) or (2.5) into (2.2) and interchange the order of integration and summation. Thus

$$f_{\tilde{X}}(x_1, x_2, \ldots, x_k) = (2\pi)^{-k} \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} \prod_{i<j} \frac{(-\rho_{ij})^{n_{ij}}}{n_{ij}!} \cdot$$

$$\prod_{i=1}^{k} \int_{-\infty}^{\infty} e^{-it_i x_i - \frac{1}{2}t_i^2}\, t_i^{n_i}\, dt_i. \tag{2.9}$$

Using (1.3), the integral in (2.9) is easily evaluated obtaining

$$f_{\underset{\sim}{X}}(x_1,x_2,\ldots,x_n) = \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} \prod_{i<j} \frac{(-\rho_{ij})^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k} (-1)^{n_i}$$

$$\cdot H_{n_i}(x_i)\phi(x_i).$$

The first part of the conclusion follows upon observing that $\sum_{i=1}^{k} n_i = 2 \sum_{i<j} n_{ij}$. Hence $\prod_{i=1}^{k} (-1)^{n_i} = (-1)^{\sum_{i<j} n_{ij}}$, verifying (2.1). The absolute convergence follows from (2.7), (2.8) and the observation that convergence for $|\rho_{ij}| < 1/(k-1)$ implies absolute convergence, since (2.1) is a power series.

<u>REMARKS</u>. In [9], Kibble actually ordered the terms as indicated in (1.6). This is equivalent to (2.1) whenever the series converges absolutely. Kibble did not assume $|\rho_{ij}| < 1/(k-1)$ and merely asserted that term-by-term integration is permissible and that the series would then be absolutely convergent for all values of the variables if R is positive definite.

We now reexamine the hypothesis $|\rho_{ij}| < 1/(k-1)$ to show that if this condition is violated, the Kibble series may in fact diverge.

If $k = 2$, $f_{\underset{\sim}{X}}(x_1,x_2)$ is the bivariate normal probability density function and the condition of Theorem 1 always holds. Theorem 1 is precisely Mehler's theorem in that case. Hence consider $k > 2$.

For $n \geq 0$,

$$H_n(x) = n! \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m x^{n-2m}}{2^m m!(n-2m)!} , \tag{2.10}$$

so that

$$H_n(0) = \begin{cases} 0 & , \ n \text{ odd} \\ \dfrac{n!(-1)^{n/2}}{2^{n/2}(n/2)!} & , \ n \text{ even} . \end{cases} \tag{2.11}$$

We will now calculate $f_{\underset{\sim}{X}}(0,0,\ldots,0)$, using the Kibble series, for the special case $\rho_{ij} = \rho$, $i \neq j$, and compare this with (1.1), which gives $f_{\underset{\sim}{X}}(0,0,\ldots,0) = (2\pi)^{-k/2}|R|^{-\frac{1}{2}}$.

Substituting (2.11) into (2.1) we get

$$f_{\underset{\sim}{X}}(0,0,\ldots,0) = (2\pi)^{-k/2} \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} \prod_{i=1}^{k} \frac{n_i!}{(n_i/2)! \prod_{i<j} n_{ij}!} (-\rho/2)^{\sum_{i=1}^{k} n_i/2} . \tag{2.12}$$

Thus for any ρ_0 for which (2.12) converges, it will converge absolutely for any ρ with $|\rho| < \rho_0$. For such ρ, we can rearrange the order of terms in (2.12) so that it is a power series. Further, if $\rho_{ij} = \rho$, $i \neq j$, it is easily seen that

$$|R| = (1+(k-1)\rho)(1-\rho)^{k-1}.$$

Hence, the power series expansion for $(2\pi)^{-k/2}|R|^{-\frac{1}{2}}$ must be the same as the power series indicated by (2.12). However, the power series for $|R|^{-\frac{1}{2}}$ has a radius of convergence of $1/(k-1)$ which implies the divergence of (2.1) whenever $|\rho_{ij}| > 1/(k-1)$, $1 \leq i < j \leq k$. Thus, we have shown that in the equicorrelated case, Theorem 1 can not be improved upon.

3. The Convergence of the Tetrachoric Series

Kendall [6] obtained the tetrachoric series by integrating the characteristic function term-by-term as in (2.9). Then he integrated the resulting expression (2.1) term-by-term obtaining

$$\int_{h_k}^{\infty} \cdots \int_{h_1}^{\infty} f_{\underset{\sim}{X}}(x_1,x_2,\ldots,x_k)dx_1dx_2\ldots dx_k = T_k(h_1,h_2,\ldots,h_k). \tag{3.1}$$

We now discuss the convergence of the tetrachoric series.

THEOREM 2. Let $h_1,h_2,\ldots h_k$ be arbitrary real numbers. Then if $|\rho_{ij}| < 1/(k-1)$, $1 \leq i < j \leq k$, the tetrachoric series $T_k(h_1,h_2,\ldots,h_k)$ converges absolutely.

Proof. From Erdélyi, Magnus, Oberhettinger, Tricomi [3], p. 208, for $n \geq 0$

$$|H_n(x)| \leq ce^{x^2/4}(n!)^{\frac{1}{2}}, \tag{3.2}$$

where $c \sim 1.086$. Then, from (2.1),

$$|f_{\underset{\sim}{X}}(x_1,x_2,\ldots,x_k)| \leq c^k \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} \prod_{i<j}\left(\frac{|\rho_{ij}|^{n_{ij}}}{n_{ij}!}\right)e^{\sum_{i=1}^{k} x_i^2/4} \prod_{i=1}^{k}(n_i!)^{\frac{1}{2}}\phi(x_i)$$

$$= \frac{c^k}{(2\pi)^{k/2}} e^{-\sum_{i=1}^{k} x_i^2/4} \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} \prod_{i<j} \frac{|\rho_{ij}|^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k}(n_i!)^{\frac{1}{2}}. \tag{3.3}$$

Let $\bar{\rho} = \max_{i<j} |\rho_{ij}| < 1/(k-1)$. Then,

$$\prod_{i<j} \frac{|\rho_{ij}|^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k} (n_i!)^{\frac{1}{2}} = \left(\prod_{i<j} \frac{|\rho_{ij}|^{n_{ij}}}{n_{ij}!} \prod_{i>j} \frac{|\rho_{ji}|^{n_{ji}}}{n_{ji}!} \prod_{i=1}^{k} n_i! \right)^{\frac{1}{2}}$$

$$= \left\{ \prod_{i=1}^{k} \left[n_i! \prod_{j:i<j} \frac{|\rho_{ij}|^{n_{ij}}}{n_{ij}!} \prod_{j:i>j} \frac{|\rho_{ji}|^{n_{ji}}}{n_{ji}!} \right] \right\}^{\frac{1}{2}}.$$

The expression inside the bracket is a term of the multinomial expansion of $\left(\sum_{j:i<j} |\rho_{ij}| + \sum_{j:i>j} |\rho_{ij}| \right)^{n_i}$. Thus

$$\prod_{i<j} \frac{|\rho_{ij}|^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k} (n_i!)^{\frac{1}{2}} \leq \left(\sum_{j:i<j} |\rho_{ij}| + \sum_{j:i>j} |\rho_{ji}| \right)^{\Sigma n_i/2} \leq ((k-1)\bar{\rho})^{\Sigma_{i<j} n_{ij}},$$

since $\sum_{i=1}^{k} n_i = 2 \sum_{i<j} n_{ij}$. Substituting this into (3.3), we have

$$|f_{\tilde{X}}(x_1,x_2,\ldots,x_k)| \leq \frac{c^k e^{-\sum_{i=1}^{k} x_i^2/4}}{(2\pi)^{k/2}} \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} ((k-1)\bar{\rho})^{\sum_{i<j} n_{ij}}$$

$$= \frac{c^k e^{-\sum_{i=1}^{k} x_i^2/4}}{(2\pi)^{k/2}} \prod_{i<j} \sum_{n_{ij}=0}^{\infty} ((k-1)\bar{\rho})^{n_{ij}}$$

$$= \frac{c^k e^{-\sum_{i=1}^{k} x_i^2/4}}{(2\pi)^{k/2}} (1-(k-1)\bar{\rho})^{-k(k-1)/2}. \qquad (3.4)$$

Since (3.4) dominates all partial sums of the Kibble series and is obviously integrable, the dominated convergence theorem applies and the Kibble series is integrable for $|\rho_{ij}| < 1/(k-1)$. Hence,

$$P\{X_1>h_1,\ldots,X_k>h_k\} = \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} \prod_{i<j} \frac{\rho_{ij}^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k} \int_{h_i}^{\infty} H_{n_i}(x_i)\phi(x_i)dx_i.$$

For $n_i > 0$, from (1.3)

$$\int_{h_i}^{\infty} H_{n_i}(x)\phi(x)dx = H_{n_i-1}(h_i)\phi(h_i) \qquad (3.5)$$

and for $n_i = 0$, since $H_o(x) = 1$, we get

$$\int_h^\infty H_o(x)\phi(x)dx = \int_h^\infty \phi(x)dx = 1-\Phi(h) = \left[\frac{1-\Phi(h)}{\phi(h)}\right]\phi(h),$$

as required by (1.2), and (3.1) follows.

To show the absolute convergence of $T_k(h_1,h_2,\ldots h_k)$, we first note that for $n = 0$

$$|H_{-1}(x)| \leq 3e^{x^2/2}$$

and for $n > 0$

$$|H_{n-1}(x)| \leq ce^{x^2/4}((n-1)!)^{\frac{1}{2}} \leq 3(n!)^{\frac{1}{2}}e^{x^2/2}.$$

Consequently, by a minor modification of the argument leading to (3.4)

$$|T_k(h_1,h_2,\ldots,h_k)| \leq \frac{3^k}{(2\pi)^{k/2}}(1-(k-1)\bar{\rho})^{-k(k-1)/2},$$

for $|\rho_{ij}| < 1/(k-1)$, establishing the absolute convergence.

We now investigate the possible relaxation of the hypothesis $|\rho_{ij}| < 1/(k-1)$, $1 \leq i < j \leq k$. For this purpose, we first investigate the computation of the orthant probabilities for k = 3 using the Kibble ordering. Thus, we compute

$$(2\pi)^{-3/2} \sum_{N=0}^{\infty} \sum \frac{\rho_{12}^{n_{12}} \rho_{13}^{n_{13}} \rho_{23}^{n_{23}}}{n_{12}! n_{13}! n_{23}!} H_{n_{12}+n_{13}-1}(0) H_{n_{12}+n_{23}-1}(0) H_{n_{13}+n_{23}-1}(0).$$

From (2.11), in order that the product $\prod_{i=1}^{k} H_{n_{i-1}}(0)$ not vanish, we must have every $n_i = 0$ or an odd positive integer. We show that, for k = 3, the only non-vanishing terms are $n_{12} = n_{13} = n_{23} = 0$ or two of n_{12}, n_{13}, n_{23} are zero and the remaining one is odd. To see this, note that if one of n_{12}, n_{13}, n_{23} is zero, say n_{12}, then n_{13} and n_{23} are odd and $n_{13} + n_{23}$ is even and the term vanishes. If none are zero and $n_{12} + n_{13}$ is odd, then either n_{12} or n_{13} must be even. Assume n_{12} is even, then n_{23} is odd and $n_{13} + n_{23}$ is even, resulting in a vanishing term. Consequently, for k = 3, the non-vanishing terms are (0,0,0), (1,0,0), (3,0,0),... and their permutations.

Therefore

$$T_3 = (\tfrac{1}{2})^3 + (2\pi)^{-3/2} \sum_{m=0}^{\infty} \left[\frac{\rho_{12}^{2m+1}+\rho_{13}^{2m+1}+\rho_{23}^{2m+1}}{(2m+1)!} H_{2m}^2(0)H_{-1}(0) \right]$$

$$= \frac{1}{8} + \frac{1}{4\pi} \sum_{m=0}^{\infty} \left[\frac{\rho_{12}^{2m+1}+\rho_{13}^{2m+1}+\rho_{23}^{2m+1}}{2m+1} \frac{(2m)!}{2^{2m}(m!)^2} \right]$$

$$= \frac{1}{2} - \frac{(\cos^{-1}\rho_{12}+\cos^{-1}\rho_{13}+\cos^{-1}\rho_{23})}{4\pi} ,$$

the well-known formula for the trivariate orthant probability; also the power series for $\cos^{-1}x$ converges for $|x| < 1$. Hence, despite the divergence of the series for the density function at 0 for $\rho > \tfrac{1}{2}$, the tetrachoric series for the trivariate orthant probabilities converges for all ρ_{ij}.

The analysis of the convergence of the tetrachoric series for orthant probabilities for $k > 3$ is somewhat more complicated. To aid in this analysis, we introduce the following lemma.

LEMMA 1. If $k \geq 2$ is even, then the non-zero terms in (1.5), when $h_i = 0$, $i = 1,2,\ldots,k$, are those terms in which an even number (including zero) of the n_i's are zero.

If $k \geq 3$ is odd, a term in (1.5) is non-zero, when $h_i = 0$, $i = 1,2,\ldots,k$, if and only if an odd number of the n_i's are zero.

Proof. From (2.11), (1.5) and (1.2), a term in the tetrachoric series for the orthant probabilities is non-zero if and only if each n_i is either zero or a positive odd integer.

Considering such terms, if $k \geq 2$ is even and an odd number of the n_i's are zero, then $\sum_{i=1}^{k} n_i = 2\Sigma n_{ij}$ is even and is the sum of an odd number of positive odd integers, which is a contradiction.

Similarly, if $k \geq 3$ is odd and an even number of the n_i's are zero, then $\sum_{i=1}^{k} n_i$ is the sum of an odd number of odd numbers and can not be even.

We now investigate the convergence of the tetrachoric series for $k > 3$.

LEMMA 2. Let $k \geq 4$ be an even integer and let

$n_{12} = n_{34} = \ldots = n_{k-1,k} = 2m+1$, $n_{ij} = 2m$ otherwise, $m = 0,1,2,\ldots,$.

Then if $\rho > 1/(k-1)$,

$$M_{m,k}(\rho) = \frac{\rho^N}{\prod_{i<j} n_{ij}!} \left(\prod_{i=1}^{k} H_{n_i-1}(0)\right) (\phi(0))^k \tag{3.6}$$

tends to infinity as $m \to \infty$, where $N = \sum_{i<j} n_{ij}$.

Proof. Since $N = \frac{k}{2}(2m(k-1)+1)$,

$$M_{m,k}(\rho) = \frac{\rho^{\frac{k}{2}(2m(k-1)+1)}}{((2m+1)!)^{k/2}((2m)!)^{k(k-2)/2}} (H_{2m(k-1)}(0))^k(\phi(0))^k.$$

Write

$$M_{m,k}(\rho) = \frac{\rho^{\frac{k}{2}(2m(k-1)+1)}}{(2m+1)^{k/2}(2m)!^{k(k-1)/2}} (H_{2m(k-1)}(0))^k(\phi(0))^k.$$

Then, using (2.11) and the elementary inequality

$$(2\pi)^{\frac{1}{2}}e^{-n}n^{n+\frac{1}{2}} < n! < 2(2\pi)^{\frac{1}{2}}e^{-n}n^{n+\frac{1}{2}} , \qquad (3.7)$$

we get

$$M_{m,k}(\rho) > \frac{(2\pi)^{-k(k+1)/4}\rho^{k/2(2m(k-1)+1)}(k-1)^{mk(k-1)}}{(2m+1)^{k/2}m^{k(k-1)/4}\,2^{k(3k-1)/4}}$$

$$= \frac{(2\pi)^{-k(k+1)/4}\rho^{k/2}}{(2m+1)^{k/2}\,m^{k(k-1)/4}\,2^{k(3k-1)/4}}\cdot(\rho(k-1))^{mk(k-1)}, \tag{3.8}$$

which clearly tends to infinity, whenever $\rho > 1/(k-1)$. This leads to the following therorem.

THEOREM 3. The tetrachoric series (1.5) with $h_i = 0$, $i = 1,2,\ldots,k$, diverges for k an even integer whenever $\rho_{ij} > \rho > 1/(k-1)$, $1 \leq i < j \leq k$.

Proof. The proof is immediate upon observing that for each fixed m, $M_{m,k}(\rho)$ is a term in the tetrachoric series for orthant probabilities when $\rho_{ij} = \rho$, $1 \leq i < j \leq k$. Thus we have exhibited a divergent subsequence of terms, which clearly precludes convergence. The same lower bound (3.8) applies if $\rho_{ij} > \rho$, $1 \leq i < j \leq k$, yielding the same conclusion in this case.

We now consider the tetrachoric series with Kibble ordering for the orthant probabilities and even values of k, restricting to the case $\rho_{ij} = \rho$, $1 \leq i < j \leq k$. This is a power series in ρ, of the form

$\sum_{N=0}^{\infty} \alpha_{N,k}\rho^N$ and the previous argument does not show that $\alpha_{N,k}$ does not tend to zero, since

$$\alpha_{N,k} = \sum_{\substack{n_{ij} \geq 0 \\ \sum n_{ij} = N}} \prod_{i<j} (n_{ij}!)^{-1} (\phi(0))^k \prod_{i=1}^{k} H_{n_i - 1}(0) \tag{3.9}$$

and in Lemma 2 we made a specific choice of the n_{ij}'s, rather than computing $\alpha_{N,k}$.

THEOREM 4. The tetrachoric series with Kibble ordering (1.6) for orthant probabilities diverges for even $k \geq 4$, whenever $\rho_{ij} \geq \rho > 1/(k-1)$, $1 \leq i < j \leq k$.

Proof. Let $N = \frac{k}{2}(2m(k-1)+1)$. The $M_{m,k}(\rho)$ (3.6) is a term in (3.9) for that N. From Lemma 1, the non-zero terms of (3.9) all have an even number of n_i's equal to zero.

If all n_i are odd, then from (2.11), the sign of such terms is given by

$$(-1)^{(\Sigma n_i - k)/2} = (-1)^{N-k/2}$$

and depends only on N. Thus, for fixed N, all terms with the opposite sign must be among those terms with a positive even number of n_i's equal to zero. Let $\gamma_{N,k}$ be the sum of all terms in (3.9) with no n_i's zero and let $\beta_{N,k}$ be the sum of all terms in (3.9) with at least two n_i's equal to zero. Then, for $\rho \geq 0$,

$$0 \leq M_{m,k}(\rho) \leq \gamma_{N,k}(\rho)\rho^N, \tag{3.10}$$

where $\gamma_{N,k}(\rho) > 0$, since $N = \frac{k}{2}[2m(k-1)+1]$ and $N - \frac{k}{2}$ is necessarily even. Obviously, $\alpha_{N,k} = \gamma_{N,k} + \beta_{N,k}$.

Consider the tetrachoric series with Kibble ordering for $\ell = k - 2$. Since each term in $\beta_{N,k}$ has at least two $n_i = 0$, each such term appears in the tetrachoric series with Kibble ordering for ℓ variables with the omission of the factors $(H_{-1}(0)\phi(0))^2$. Thus,

$$\sum_{N=0}^{\infty} |\beta_{N,k}||\rho|^N \leq \phi^2(0)H_{-1}^2(0)\binom{k}{2} \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{\ell-1,\ell}=0}^{\infty} \phi^{\ell}(0) \prod_{i<j} \frac{|\rho|^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{\ell} |H_{n_i-1}(0)| .$$

Since for $1/(k-1) < \rho < 1/(k-3)$, $\sum \beta_{N,k}\rho^N$ converges absolutely by Theorem 2, $\beta_{N,k}\rho^N \to 0$ as $N \to \infty$ and $M_{m,k}(\rho) \to \infty$. Further, (3.10) implies that $\sum_{N=0}^{\infty} \alpha_{N,k}\rho^N$ diverges for $1/(k-1) < \rho < 1/(k-3)$, but this is a power series and hence divergence is established for all $\rho > 1/(k-1)$. The conclusion now follows trivially.

There is an extensive literature on the evaluation of orthant probabilities for $k = 4$. Many references may be found in N.L. Johnson and S. Kotz [5], pages 53-58. One of these methods merits comment here since it specifically employs the tetrachoric series with Kibble ordering. In P.A.P. Moran [11], the tetravariate orthant probability is used to calculate the variance of the Spearman rank correlation coefficient. To obtain the orthant probability, Moran rederived the tetrachoric series and explicitly calculated the first few terms of the tetrachoric series with Kibble ordering. He made no comment about the validity of the series, but did comment that for the problem at hand calculation is tedious.

The convergence properties of the tetrachoric series for orthant probabilities appear to be more difficult to investigate when k is odd. These properties are discussed in the following theorems.

THEOREM 5. Let k be an odd integer ≥ 3 and order all the correlation coefficients by absolute magnitude, that is, $|\rho_1| \geq |\rho_2| \geq \ldots \geq |\rho_q|$, $q = \binom{k}{2}$, so that ρ_ν has the νth largest magnitude among the ρ_{ij}, $1 \leq i < j \leq k$. If the tetrachoric series (tetrachoric series with Kibble ordering) for orthant probabilities converges absolutely for $\ell = k-1$ with correlation coefficients ρ_ν, $\nu = 1,2,\ldots,\binom{\ell}{2}$, then the tetrachoric series (tetrachoric series with Kibble ordering) for the orthant probabilities converges absolutely for k.

Proof. We have

$$T_k \leq \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{k-1,k}=0}^{\infty} (\phi(0))^k \prod_{i<j} \frac{|\rho_{ij}|^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{k} |H_{n_i-1}(0)|. \tag{3.11}$$

From Lemma 1, every non-zero term in T_k must have at least one n_i equal to zero. Thus, successively setting each n_i equal to zero, we get

$$T_k \leq H_{-1}(0) \sum_{\nu=1}^{k} \sum_{\substack{n_{ij}\geq 0 \\ i,j\neq\nu}} (\phi(0))^k \prod_{\substack{i<j \\ i,j\neq\nu}} \frac{|\rho_{ij}|^{n_{ij}}}{n_{ij}!} \prod_{\substack{i=1 \\ i\neq\nu}}^{k} |H_{n_i-1}(0)|; \tag{3.12}$$

the right side of (3.12) dominates (3.11) since we have enumerated terms with more than one $n_i = 0$ more than once. For each fixed ν, the inner sum dominates $\phi(0)T_{k-1}$. Thus (3.11) converges absolutely whenever each of the k series in (3.12) converges absolutely. Each such series depends on $\binom{\ell}{2}$ correlation coefficients. For each series in (3.12) if ρ_{ij} is not in the set of $\binom{\ell}{2}$ correlation coefficients of largest magnitude, we replace it by one that does not appear in the series in such a way that each ρ_ν, $\nu = 1,2,\ldots,\binom{\ell}{2}$ appears once. Thus,

$$T_k \leq k(\phi(0))^k H_{-1}(0) \sum_{n_{12}=0}^{\infty} \cdots \sum_{n_{\ell-1,\ell}=0}^{\infty} \prod_{i<j} \frac{|\rho_\nu|^{n_{ij}}}{n_{ij}!} \prod_{i=1}^{\ell} |H_{n_i-1}(0)|,$$

which converges by hypothesis. The conclusion for the tetrachoric series with Kibble ordering follows upon rearrangement of the terms.

REMARK. We have actually proved the absolute convergence of the tetrachoric series whenever each of the k series in (3.12) converges absolutely.

COROLLARY 1. If each $\rho_{ij} = \rho$, then the tetrachoric series (tetrachoric series with Kibble ordering) for the orthant probabilities converges absolutely whenever $-1/(k-1) < \rho < 1/(k-2)$, for k an odd integer ≥ 5.

Proof. The conclusion follows from combining Theorem 5 and Theorem 2.

COROLLARY 2. If $k = 3$, the tetrachoric series (tetrachoric series with Kibble ordering) converges absolutely.

Proof. This is immediate from Theorems 5 and 2.

The observations utilized in Theorem 5 also enable us to establish the following.

THEOREM 6. If $k \geq 5$ is odd and if $\rho_{ij} > 1/(k-2)$ for $1 \leq i < j \leq k$, then the tetrachoric series (tetrachoric series with Kibble ordering) for the orthant probabilities diverges.

Proof. Consider the tetrachoric series with Kibble ordering. From Lemma 1, the contribution to $\alpha_{N,k}$ (3.9) is zero unless an odd number of the n_i's are zero and the remainder are positive odd integers. Write $\alpha_{N,k} = \gamma_{n,k} + \beta_{N,k}$, where $\gamma_{N,k}$ are these terms with exactly one $n_i = 0$. Then

$$\gamma_{N,k} = k \sum_{\substack{n_{ij} \geq 0 \\ \sum n_{ij} = N}} \prod_{i<j} (n_{ij}!)^{-1} \phi^{\ell}(0) H_{-1}(0) \prod_{i=1}^{\ell} H_{n_i-1}(0),$$

where $\ell = k - 1$ and $1 \leq i < j \leq \ell$. From Theorem 4, $\Sigma \gamma_{N,k} \rho^N$ diverges for

$\rho > 1/(k-2) = 1/(\ell-1)$. $\beta_{N,k}$ corresponds to those terms with at least 3 or more $n_i = 0$. Each such term occurs in the tetrachoric series with Kibble ordering for $k - 3$ variables. For $1/(k-2) < \rho < 1/(k-4)$, that series is absolutely convergent. Thus, the methodology of Theorem 4 applies and we obtain the conclusion.

So far we have only considered the case of orthant probabilities. Therefore, we now show that similar results obtain in a somewhat more general case. For this purpose, we require the following lemma.

LEMMA 3. Let $x \neq s\pi/2(k-1)$, where s and k are integers, and $k \geq 2$. Then

$$\cos u_m(x) = \cos\left[\left(\frac{2(k-1)m-1}{2}\right)^{\frac{1}{2}} x - \left(\frac{(k-1)m-1}{2}\right)\pi\right]$$

does not tend to zero as m tends to infinity.

Proof. Let $m_j = 4j^2(k-1)$, $j = 1,2,\ldots,$. Then

$$\cos u_{m_j}(x) = \cos [4j^2(k-1)^2 - \tfrac{1}{2})^{\frac{1}{2}} x - 2j^2(k-1)^2\pi + \tfrac{\pi}{2}]$$

$$= -\sin [2j(k-1)(1 - \frac{1}{8j^2(k-1)^2})^{\frac{1}{2}} x] .$$

Since

$$2j(k-1) - \frac{1}{4j(k-1)} < 2j(k-1)(1 - \frac{1}{8j^2(k-1)^2})^{\frac{1}{2}} < 2j(k-1),$$

for any $\varepsilon > 0$, there is a j_0 such that for all $j > j_0$

$$\left|\sin[2j(k-1)(1 - \frac{1}{8j^2(k-1)^2})^{\frac{1}{2}} x] - \sin 2j(k-1)x\right| < \varepsilon$$

Hence if

$$\lim_{j\to\infty} \sin[2j(k-1)(1 - \frac{1}{8j^2(k-1)^2})^{\frac{1}{2}} x] = 0,$$

it follows that $x = s\pi/2(k-1)$, s an integer.

We now show that under specific circumstances, the tetrachoric series diverges for $\rho_{ij} > \rho > 1/(k-1)$.

THEOREM 7. If $\rho_{ij} > \rho > 1/(k-1)$, $T_k(x,x,\ldots,x)$ diverges whenever $x \neq s\pi/2(k-1)$, s an integer.

Proof. Let $Q_{m,\rho}(x)$ be the term of $T_k(x,x,\ldots,x)$ with $\rho_{ij} = \rho$, $n_{ij} = m$. Then

$$Q_{m,\rho}(x) = \frac{\rho^{mk(k-1)/2}}{(m!)^{k(k-1)/2}} \phi^k(x)\, H^k_{(k-1)m-1}(x). \tag{3.14}$$

We examine the behavior of $Q_{m,\rho}(x)$ as $m \to \infty$. From Erdélyi, Magnus, Oberhettinger, Tricomi [3], p. 201,

$$H_n(x) = \frac{\Gamma(n+1)}{2^{n/2}\Gamma(\frac{n}{2}+1)} \cos\left(\left(\frac{2n+1}{2}\right)^{\frac{1}{2}}x - \frac{n\pi}{2}\right)e^{x^2/4} + O\left(\left(\frac{n}{e}\right)^{\frac{n}{2}} n^{-\frac{1}{2}}\right), \tag{3.15}$$

as $n \to \infty$. Replacing the gamma functions in (3.15) by Stirling's approximation, we obtain

$$H_n^k(x) = \frac{n^{\frac{nk}{2}}2^{k/2}}{e^{\frac{nk}{2}}} \cos^k\left(\left(\frac{2n+1}{2}\right)^{\frac{1}{2}}x - \frac{n\pi}{2}\right)e^{kx^2/4}(1+O(n^{-\frac{1}{2}})). \tag{3.16}$$

Substituting (3.16) into (3.14) and again employing Stirling's approximation yields

$$Q_{m,\rho}(x) = \rho\frac{^{mk(k-1)/2}\ 2^{k/2}\ e^{-kx^2/4}[(k-1)m-1]^{k((k-1)m-1)/2}}{m^{(m+\frac{1}{2})k(k-1)/2}e^{-k/2}(2\pi)^{k(k+1)/4}} \cdot$$

$$\cdot \cos^k\left(\left(\frac{2(k-1)m-1}{2}\right)^{\frac{1}{2}} x - \frac{((k-1)m-1)\pi}{2}\right)(1+O(m^{-\frac{1}{2}})). \tag{3.17}$$

Writing

$$((k-1)m-1)^{k(k-1)m/2} = [(k-1)m]^{k(k-1)m/2}\left(1 - \frac{1}{(k-1)m}\right)^{k(k-1)m/2}$$

$$= ((k-1)m)^{k(k-1)m/2}e^{-k/2}(1+O(m^{-1}))$$

and substituting this into (3.17), we get

$$Q_{m,\rho}(x) = \frac{((k-1)\rho)^{mk(k-1)/2} 2^{k/2} e^{-kx^2/4} [(k-1)m-1]^{-k/2}}{m^{k(k-1)/4} (2\pi)^{k(k+1)/4}} \cdot$$

$$\cos^k[(\frac{2(k-1)m-1}{2})^{\frac{1}{2}} x - (\frac{(k-1)m-1}{2})\pi)(1+O(m^{-\frac{1}{2}})). \qquad (3.18)$$

Then, if $x \neq s\pi/2(k-1)$, s an integer, it follows from Lemma 3 that there exists a subsequence $\{m_\nu\}$ such that

$$\lim_{\nu\to\infty} |Q_{m_\nu,\rho}(x)| = \infty,$$

whenever $\rho > 1/(k-1)$. Clearly, this precludes the convergence of the tetrachoric series for $\rho_{ij} = \rho > 1/(k-1)$, $1 \leq i < j \leq k$ and consequently for $\rho_{ij} > \rho$, $1 \leq i < j \leq k$.

CONCLUDING REMARKS. In this paper, we have shown that the tetrachoric series need not converge. We have exhibited some instances of convergence and some of divergence. The results of this paper do not exhaust all the possibilities, that is, not all the possible choices of the ρ_{ij}, $1 \leq i < j \leq k$ or h_i, $i = 1,2,\ldots,k$ have been treated. Further investigation will be needed to resolve the remaining cases. However, we have shown that the tetrachoric series should be employed in applications only under highly restrictive conditions.

REFERENCES

[1] ANDERSON, T.W. (1958). Introduction to Multivariate Statistical Analysis. Wiley, New York.

[2] BARLOW, R.E., BARTHOLOMEW, D.J., BREMNER, J.M., AND BRUNK, H.D. (1972). Statistical Inference under Order Restrictions. Wiley, New York.

[3] ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F., AND TRICOMI, F.G. (1953). Higher Transcendental Functions, Volume II. Mc-Graw Hill, New York.

[4] GUPTA, S.S. (1963). Probability integrals of multivariate normal and multivariate t. Ann. Math. Statist., 34, 792-828.

[5] JOHNSON, N.L. AND KOTZ, S. (1972). Distributions in Statistics, Continuous Multivariate Distributions. Wiley, New York.

[6] KENDALL, M.G. (1941). Proof of relations connected with tetrachoric series and its generalization. Biometrika, 32, 196-198.

[7] KENDALL, M.G. (1945). On the analysis of oscillating time-series. J. Roy. Statist. Soc., 108, 93-129.

[8] Kendall, M.G., and Stuart, A. (1969). The Advanced Theory of Statistics, Volume I. Griffin, London.

[9] Kibble, W.F. (1945). An extension of a theory of Mehler on Hermite polynomials. Proc. Cambridge Philos. Soc., 41, 12-15.

[10] Mehler, F.G. (1866). Über die Entwicklung einer Funktion von beliebig vielen Variabeln nach Laplace'schen Funktionen höherer Ordnung. J. Reine Angew. Math., 66, 161-176.

[11] Moran, P.A.P. (1948). Rank correlation and product moment correlation. Biometrika, 35, 203-206.

[12] Plackett, R.L. (1954). A reduction formula for normal multivariate integrals. Biometrika, 41, 351-360.

[13] Steck, G.P. (1958). A table for computing trivariate normal probabilities. Ann. Math. Statist., 29, 780-800.

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The tetrachoric series is a technique for evaluating multivariate normal probabilities frequently cited in the statistical literature. In this paper we have examined the convergence properties of the tetrachoric series and have established the following.

For orthant probabilities, the tetrachoric series converges if $|\rho_{ij}| < 1/(k-1)$, $1 \le i < j \le k$, where ρ_{ij} are correlation coefficients of a k-variate normal distributation. The tetrachoric series for orthant probabilities diverges whenever k is even and $\rho_{ij} > 1/(k-1)$ or k is odd and $\rho_{ij} > 1/(k-2)$, $1 \le i < j \le k$. (continued)

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Abstract - continued

Other specific results concerning the convergence or divergence of this series are also given.

The principal point is that the assertion that the tetrachoric series converges for all $k \geq 2$ and all ρ_{ij} such that the correlation matrix is positive definite is false.